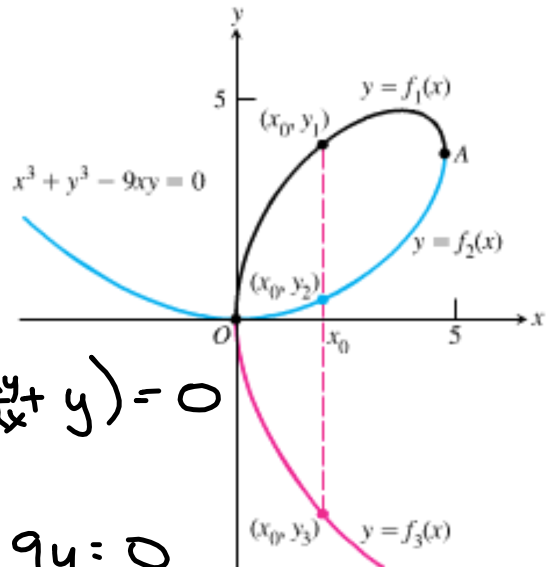
$x^3 + y^3 - 9xy = 0$ (Figure 3.47)



$$\frac{dy}{dx}: \quad 3x^2 + 3y^2\frac{dy}{dx} - 9\left(x\frac{dy}{dx} + y\right) = 0$$

$$3x^2 + 3y^2\frac{dy}{dx} - 9x\frac{dy}{dx} - 9y = 0$$

implicit differentiation.

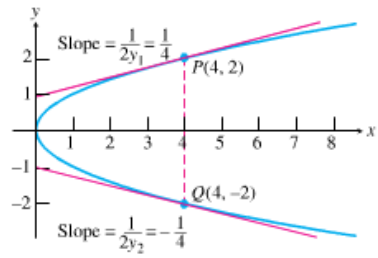
$$3y^2\frac{dy}{dx} - 9x\frac{dy}{dx} = 9y - 3x^2$$

$$\frac{dy}{dx}\left(3y^2 - 9x\right) = 9y - 3x^2$$

$$\frac{dy}{dx} = \frac{9y - 3x^2}{3y^2 - 9x}$$

EXAMPLE 1 Differentiating Implicitly

Find dy/dx if $y^2 = x$.



$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2) \cdot \frac{dy}{dx}$$

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2) \cdot \frac{dy}{dx}$$
$$= 2y\frac{dy}{dx}$$

$$y^2 = x$$

$$2y\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{2y}$$

EXAMPLE 2 Finding Slope on a Circle

Find the slope of the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

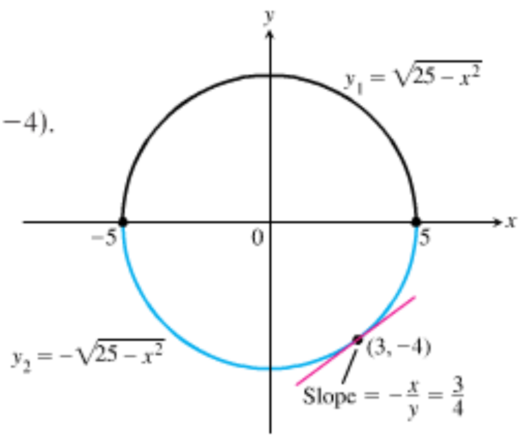


Figure 3.49 The circle combines the graphs of two functions. The graph of y_2 is the lower semicircle and passes through $(3, -4)$. (Example 2)

$$y^2 = 25 - x^2$$

$$y = \pm\sqrt{25 - x^2}$$

$$y = -\sqrt{25 - x^2}$$

$$= -(25 - x^2)^{1/2}$$

$$= -\tfrac{1}{2}(25 - x^2)^{-1/2}(-2x)$$

$$= \frac{x}{\sqrt{25 - x^2}} = \frac{3}{4}$$

or

$$x^2 + y^2 = 25$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-x}{y} = \frac{-(3)}{(-4)} = \frac{3}{4}$$

EXAMPLE 3 Solving for dy/dx

Show that the slope dy/dx is defined at every point on the graph of $2y = x^2 + \sin y$.

$\frac{dy}{dx}$: $2\frac{dy}{dx} = 2x + \cos y \frac{dy}{dx}$

$2\frac{dy}{dx} - \cos y \frac{dy}{dx} = 2x$

$\frac{dy}{dx}(2 - \cos y) = 2x$

$\frac{dy}{dx} = \frac{2x}{2 - \cos y}$ → only undefined at

$2 - \cos y = 0$

$\cos y = 2$

$\cos y$ can never $= 2$

because it can at most $= 1$

Implicit Differentiation Process

1. Differentiate both sides of the equation with respect to x.
2. Collect the terms with dy/dx on one side of the equation.
3. Factor out dy/dx.
4. Solve for dy/dx.

EXAMPLE 4 **Tangent and normal to an ellipse**

Find the tangent and normal to the ellipse $x^2 - xy + y^2 = 7$ at the point $(-1, 2)$. (See Figure 3.51.)

tangent → slope: $2x - \left(x\frac{dy}{dx} + y\right) + 2y\frac{dy}{dx} = 0$

point: $(-1,2)$

$2x - x\frac{dy}{dx} - y + 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx}(2y - x) = y - 2x$

$\frac{dy}{dx} = \frac{y-2x}{2y-x}$ $\qquad \frac{2-2(-1)}{2(2)-(-1)} = \frac{4}{5}$

$y - 2 = \frac{4}{5}(x+1)$

normal → slope: $-\frac{5}{4}$

point: $(-1,2)$ $\qquad y-2 = -5/4\,(x+1)$

EXAMPLE 5 **Finding a Second Derivative Implicitly**

Find d^2y/dx^2 if $2x^3 - 3y^2 = 8$.

$$\frac{dy}{dx}: \quad 6x^2 - 6y\frac{dy}{dx} = 0$$

$$-6y\frac{dy}{dx} = -6x^2$$

$$\frac{dy}{dx} = \frac{-6x^2}{-6y}$$

$$\frac{dy}{dx} = \frac{x^2}{y}$$

$$\frac{d^2y}{dx^2}: \quad \frac{y(2x) - x^2\frac{dy}{dx}}{y^2}$$

$$= \frac{2xy - x^2\left(\frac{x^2}{y}\right)}{y^2}$$

$$= \frac{2xy - \frac{x^4}{y}}{y^2}$$

$$= \frac{2xy}{y^2} - \frac{x^4}{y}\cdot\frac{1}{y^2}$$

$$= \frac{2x}{y} - \frac{x^4}{y^3}$$

Rational Powers of Differentiable Functions

RULE 9 Power Rule for Rational Powers of *x*

If n is any rational number, then

$$\frac{d}{dx}x^n = nx^{n-1}.$$

If $n < 1$, then the derivative does not exist at $x = 0$.

(a) $\frac{d}{dx}(\sqrt{x})$ $= x^{1/2} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

(b) $\frac{d}{dx}(x^{2/3})$ $\frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$

(c) $\frac{d}{dx}(\cos x)^{-1/5}$

$= -\frac{1}{5}(\cos x)^{-6/5}(-\sin x)$

$= \frac{1}{5}\sin x\,(\cos x)^{-6/5}$